

# Analysis of Cross Correlation between Prediction and Observation Errors of an Inertial Navigation System

Bingbing Liu and Martin Adams

**Abstract**—This paper investigates an approach to quantify the problem of cross correlation between the prediction and observation noise of an inertial navigation system (INS), which utilizes a linear Kalman filter (KF). Cross correlation is shown being introduced by use of the transformation matrix to transform body frame velocity observations into navigation frame. The effect of the cross-correlation term on the error covariance matrix and subsequently on the convergence of the filter is evaluated theoretically. With the cross-correlation term being formulated from the prediction and observation noise, it is incorporated into the KF and thus the relevant filter equations have been updated accordingly. The theoretical formulation and numerical simulations present the importance of incorporating this term into the filter. If this term was ignored, the error covariance estimates associated with the positional estimates would be too small and the filter would be “over confident”.

## I. INTRODUCTION

Inertial measurement units (IMUs) are ideal for outdoor localization applications for mobile robots because they can provide pose estimation in 3D due to their triad of orthogonal accelerometers and gyroscopes. IMUs are usually combined with external sensors to bound the accumulated errors. Under the consideration that commonly adopted GPS is unavailable, other methods to bound the errors of INS exist, e.g. [1] and [2]. In [2], a multi-aided INS was proposed to fuse IMU data with aiding information from odometry, an accurate gyro and vehicle constraints. In [1], [2] and all the other similar work that needs transforming body frame observations into navigation frame, a problem of cross correlation between the INS prediction and observation errors will occur.

Take [2] as an example, the body velocity has to be pre-multiplied with the frame transformation matrix. Since the transformation matrix evolves with three Euler angles, the errors of which are part of the state vector, error in the observation (navigation velocity) is *NOT* strictly independent of errors in the state vector. This cross correlation between the observation errors and system errors violate the assumption of the Kalman filter (KF) and could lead to filter divergence. Instead of using a tradition shaping filter ([3], [4]) to address this cross correlation problem, a new method will be proposed in this paper by adding the cross correlation term into the KF. By running the filter with and without the cross-correlation term and comparing the error covariance of each individual state variable, a conclusion will be made as whether this term can be reasonably ignored...

Bingbing Liu is with Data Storage Institute, A\*STAR, Singapore  
LIU.Bingbing@dsi.a-star.edu.sg

Martin Adams is with School of Electrical and Electronic Engineering, Nanyang Technological University eadams@ntu.edu.sg

## II. A MULTI-AIDED INERTIAL NAVIGATION SYSTEM

In [2], an INS is formulated to fuse inertial measurements with multi-aiding information. A linear KF is used to combine the inertial and multi-aiding data. The state vector of the KF is  $X = [\delta P_n^T, \delta V_n^T, \delta \Psi_n^T]^T$ , where,  $\delta P_n$ ,  $\delta V_n$  and  $\delta \Psi_n$  are navigation frame position, velocity and attitude error vectors of the INS and  $\Psi_n$  consists of  $\gamma, \beta, \theta$ , which are Euler angles (yaw, pitch and roll). The Pinson error model is used to provide predictions of  $X$ . The prediction error  $\omega$  is assumed as a Gaussian white noise with covariance  $Q$ , i.e.,  $E[\omega(k)\omega(j)^T] = Q\delta_{kj}$ , with  $\delta$  being the Dirac delta function.

The velocity observation of the INS can be made by transforming the body frame velocities into navigation frame,

$$z_v^{aiding}(k) = C_b^n(k) [v_x(k) \ v_y(k) \ v_z(k)]^T. \quad (1)$$

where  $z_v^{aiding}(k)$  represents the velocity observations from the aiding method and  $C_b^n(k)$  is the transformation matrix. Like the process noise  $\omega$ , the observation noise  $v$  is also assumed to be a zero mean Gaussian white noise with  $E[v(k)v(j)^T] = R\delta_{kj}$ . It is further assumed that the process noise and observation noise are uncorrelated. That is,  $E[\omega(k)v(j)^T] = \mathbf{0} \ \forall k, j$ . However this assumption is generally violated because  $C_b^n$  is used to transform the body frame velocity observations into the navigation frame. Since  $C_b^n$  consists of Euler angles  $\gamma, \beta, \theta$ , the system predictions and observations are correlated...

## III. ANALYSIS OF CROSS CORRELATION BETWEEN PREDICTION AND OBSERVATION ERRORS

Cross correlation problems were addressed by shaping filters, e.g., [3], [4]. In our case, the power spectral density of the colored noise corrupting the navigation velocity is difficult to correctly duplicate with any existent shaping filters. To make the problem tractable, an assumption is made that the cross correlation between the process and observation noise occurs just one time step apart. This assumption is reasonable because  $C_b^n(k)$  is part of the estimation result at time  $k$ , and at  $k+1$ , it is used to transform velocity observation from the body frame to navigation frame.

In the following model, the cross-correlation term between the process noise and observation noise will be introduced in the KF to study its effect.

$$X(k) = FX(k-1) + \omega(k) \quad (2)$$

$$z(k) = HX(k) + v(k) \quad (3)$$

$$E[\omega(k)v(j)^T] = U\delta_{k,j-1} \quad (4)$$

Equations 2 to 3 represent a system with an assumed time-variant process and observation model with independent zero

mean, white Gaussian noise, which is consistent with the proposed multi-aided INS. Equation 4 denotes that the cross correlation occurs only between  $\omega(k)$  and  $v(k+1)$ . The KF is thus updated accordingly to include this cross correlation item. It is interesting to note that the correlation between the process and measurement noise has no effect on the estimated state vector itself. However, the updated error covariance of the state at  $k$  becomes

$$\begin{aligned}
& P(k|k) \\
&= P(k|k-1) - [P(k|k-1)H^T + U]S(k)^{-1}[P(k|k-1)H^T \\
&+ U]^T \\
&= P(k|k-1) - P(k|k-1)H^T S(k)^{-1}HP(k|k-1)^T \\
&+ P(k|k-1)H^T S(k)^{-1}U^T + US(k)^{-1}HP(k|k-1)^T \\
&+ US(k)^{-1}U^T \\
&= P(k|k-1) - W(k)S(k)W(k) + P(k|k-1)H^T S(k)^{-1}U^T \\
&+ US(k)^{-1}HP(k|k-1)^T + US(k)^{-1}U^T
\end{aligned} \tag{5}$$

Compared with a standard linear KF, there are three more terms in equation 5 involving  $U$ , meaning that certain elements of the covariance matrix  $P(k|k)$  are affected. As symbolic solution of each element of this matrix is complex for direct use, simulations will be utilized to show the effect of this correlation using realistic numerical examples...

#### IV. SIMULATION RESULTS

A simulated data is used to demonstrate the effect of the cross correlation term on the state estimate. Basically the simulated data set used here is generated by simulating a circular arc as the vehicle's trajectory. In Section III, it was seen that the cross-correlation term  $U$  affects the modified updated covariance matrix  $P(k|k)$ . Intuitively, the effect of  $U$  is that the value of some entries of the covariance matrix  $P(k|k)$  should be increased, without which will probably lead to earlier filter divergence. It is necessary to know which elements of  $P(k|k)$  are affected and by how much, to study the effect of  $U$ . In order to obtain this information, the difference of each element of the covariance matrix estimated with and without  $U$  is shown in Figure 1. In these bar diagrams, the 2 horizontal axes define each element of the matrix and vertical axis shows how large the element is.

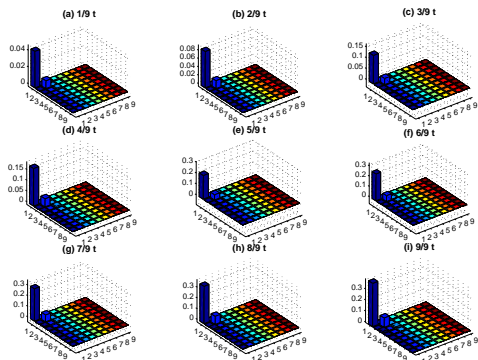


Fig. 1. 3D bar diagram of the differences between each element of the covariance matrix  $P(k|k)$  with and without the term  $U$ , (a)-(i) at 9 different time instants.

It is seen from the 9 sub-figures in Figure 1 that, except the first and second diagonal elements, the other ones of the

covariance matrix are small and hence much less affected by  $U$ . The first and second diagonal elements of the covariance matrix correspond to the error covariances of the *North* and *East* positions. Hence, the existence of  $U$  causes the positional error covariances to increase. Figure 2 shows the errors of the *North* and *East* positions with  $2\sigma$  error bounds. The  $2\sigma$  error bounds are calculated from the corresponding elements in the error covariance matrix  $P(k|k)$ , with and without  $U$ . It is seen that because of the existence  $U$ , the positional error bounds were increased and hence filter consistency resulted for a longer time, after which the positional errors grew beyond the  $2\sigma$  error bounds...

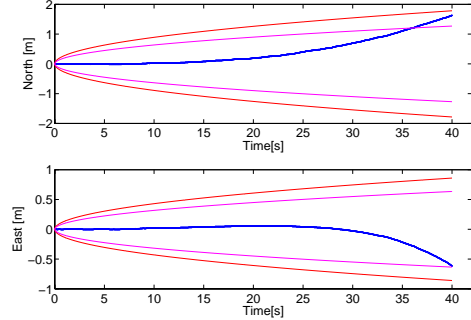


Fig. 2. The positional errors with  $2\sigma$  error bounds, with and without  $U$  in the *North* (upper graph) and *East* (lower graph) directions. In both graphs, the red curves represent  $2\sigma$  error bounds with  $U$  and the magenta ones represent those without  $U$ .

#### V. CONCLUSIONS

In this paper the cross correlation between the process and observation noise of a type of INS has been presented. The correlation term has been formulated and incorporated into the linear Kalman filter. A simulation has been generated using numerical signals to evaluate the cross-correlation term on the error covariance matrix and further on the performance of the filter. The simulated results have shown that the position error elements in the error covariance matrix have been increased since the cross-correlation term was added into the estimation process. This means that the error bounds of the position estimates have been increased, which could postpone the divergence of the filter. Without consideration of this term, the error covariance associated with the position estimates are smaller and the filter is "over confident". Simulation results further reveal that only the positional error covariances are mainly affected and other elements such as the velocity and Euler angles are significantly less affected.

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